Turbulence Modeling for Complex Shear Flows

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Nomenclature

C = chord length
$C_p, C_1, C_\mu, C_2, C_4$
$C_{\epsilon 1}, C_{\epsilon 2}, C_{\epsilon 3}, C_s, C_{\phi 1}$ = modeling constants
D, d = dissipation term, channel width
g_{ik} = metric tensor
k = turbulent kinetic energy, $= \frac{1}{2} g^{ik} \overline{u_i u_k}$
ℓ, L = length scale
p, p' = mean pressure, fluctuating pressure
P, P_{ik} = turbulence production
r = radial distance from the axis
R = radial distance normalized by the tip radius/radius of curvature
R_T = local Reynolds numbers, $= k^2/\nu\epsilon$
R_{ic} = generalized gradient Richardson number, $= -\epsilon_{ipj}\Omega^P/U_{i,j}$
S_{ij} = fluctuating strain tensor, $\frac{1}{2}(u_{i,j} + u_{j,i})$
$\bar{S}_{ij}$ = mean strain, tensor, $\frac{1}{2}(U_{i,j} + U_{j,i})$
U^i, u^i = mean and fluctuating velocity
$U^{*m}_{,\ell}$ = $U^m_{,\ell} + \epsilon^m_{p\ell}\Omega_P$
$\overline{u_i u_j}$ = Reynolds stress tensor
U, U_e = local/freestream (edge) velocity
U_s = streamwise relative velocity normalized by U_e
V_θ, V_r, V_z = tangential, radial, and axial velocity in a cylindrical coordinate system (r, θ, z)
U, V, W = mean velocities in principal(s) and transverse directions (n, r)
u, v, w = fluctuating component of velocities in principal(s) and transverse directions (n, r)
u^*, u_0^* = friction velocity $(\sqrt{\tau_w/\rho})$ with and without rotation
x^j, x_j = contravariant and covariant coordinate variables
s, n, r = streamwise, normal, and radial directions; n is distance normal to surface
x, y, z = coordinates used; x is along surface, y and z are transverse coordinates
δ = boundary-layer/shear-layer thickness
δ_{ij} = Kronecker tensor
ω = specific dissipation rate
ϵ = turbulent dissipation rate, $2\nu\overline{S_{ij}S^{ij}}$
ϵ_{ipj} = permutation tensor
μ = molecular viscosity
$\mu_{\rm eff}$ = $\mu + \mu_t$
ν_t = turbulent kinematic viscosity
τ, τ_{ij} = shear stress
ρ = density
$\sigma_k, \sigma_\epsilon, \sigma_w$ = modeling constants
Ω = angular velocity/rotation number $(\Omega d/U_m)$
Ω^P = contravariant component of angular velocity
β = angle between the axis and the streamwise direction

Subscripts

i, j, k, ℓ, m, n = indices
m, max = mean, maximum
0 = values without extra strain/edge/freestream

Introduction and Scope of the Review

MANY of the flows encountered in practice are turbulent and three-dimensional in nature. In many situations, the three-dimensionality, curvature, rotation, shock/boundary-layer interaction, bouyancy, flow separation or reversal, and other effects introduce changes in the turbulence structure, thus invalidating many of the turbulence models used widely for "simple" and "mildly complex" shear layers. It becomes increasingly important to employ more physics and/or constitutive equations in providing suitable closure models for adequate prediction of these complex flows.

In Ref. 1, Bradshaw provided a basis on which the shear flows can be classified as "simple" and "complex." The simple shear layer is one where the significant rate-of-strain component is $\partial U/\partial y$, the gradient normal to the wall in a

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two-dimensional flow. The flow subjected to extra rates of strain is called "complex shear layers." These effects are caused by additional velocity gradients, such as $\partial V/\partial x$, $\partial U/\partial x$, $\partial U/\partial z$, $\partial W/\partial z$, $\partial W/\partial y$, $\partial W/\partial x$, and the "extra" effects caused by the curvature, buoyancy, and Coriolis forces. It is known that even small values of these extra rates of strain can have a surprising effect on the turbulence structure, shear stresses, and mean velocity profiles. Structural changes have been observed experimentally when the effect of curvature, rotation, and three-dimensionality is present (e.g., Refs. 2-7). Some of the examples where complex shear layers are present are as follows (see Bradshaw in Ref. 1 for the source of experimental data for some of these flows):

1) Shear layers on rotating bodies, such as those that occur on turbomachinery rotor blades, helicopter rotors, and meteorological flows. The Coriolis forces tend to change the turbulence structure through redistribution. The boundary layers and wakes on these blades are almost invariably three-dimensional, thus introducing an additional strain due to $\partial W/\partial y$.

2) Shear layers developing on curved surfaces, such as boundary layers on turbine blades, wall jets used in cooling applications, and curved flow in turbomachinery passages. The centrifugal forces introduce additional strain; in most practical cases, the shear layers are three-dimensional, introducing additional strain due to $\partial W/\partial y$ and the centrifugal acceleration.

3) Three-dimensional boundary layers and wakes (e.g., finite wings, fuselages, marine vehicles, turbomachinery hub, and wall boundary layers).

4) Shear layers with lateral convergence and divergence, as in flow through annular diffusers/nozzles.

5) Separated flows, such as those arising from a shock/boundary-layer interaction, and separated flows on wings, cascades, rotors, etc. In these cases, an abnormal distribution of $\partial U/\partial y$, as well as strain due to other velocity gradients, introduces changes in the structure. Additional velocity gradients occurring in a separation zone of a three-dimensional boundary layer presents a formidable case for a turbulence modeler.

6) Swirling flows and vortices (e.g., tornadoes, hurricanes, turbomachinery annulus flow, etc.). In this case, the additional strain due to centrifugal forces, as well as presence of $\partial V_\theta/\partial r$, $\partial V_z/\partial r$, $\partial V_\theta/\partial z$, $\partial V_z/\partial z$, etc., brings about structural changes in the turbulence.

7) Compressibility effect. This effect and the resulting reduction in the stream tube area introduce strains in addition to those caused by $\partial V/\partial y$ and $\partial W/\partial y$. For most cases, the incompressible models provide reasonable agreement with the data.

8) Flow with other effects, such as buoyancy, chemical reaction, etc.

Some of these flows are illustrated in Fig. 1 of Ref. 8. These are a few of the many complex flows that are encountered in practice.

It is evident from the cases presented at the Stanford University Conference,[1,9] that many of the models are successful in predicting "simple flows," but are not adequate for "complex flows." They fail to capture the essential physics of the flow. The turbulence in these complex flows is usually anisotropic, nonhomogeneous, and three-dimensional. The computation of these flows requires increasingly sophisticated models, some of which are beyond the scope of present-day computers.

It is evident from many of the calculations presented[1,9] that the empirical constants used in the turbulence transport equations, which are based on two-dimensional simple flows, are invalid or inadequate for complex flows. In many instances, the predictions from the full Reynolds stress equations were as good as those obtained by simple corrections for the effects of curvature and rotation. Hence, the field of turbulence modeling for complex flows is confusing and conflicting. Intuition and ad-hoc assumptions (many of them questionable) dominate the art of turbulence modeling in "complex flows." Many of the models do predict one set of flows, but fail to capture the physics in other similar flows. The basic reason as to why these models predict some of the complex flows is not clear. It is in this setting that the author has sought to review the literature on turbulence modeling of complex flows. It is hoped that this review will point out the deficiencies and strengths of various models and, most importantly, that it will act as a catalyst for the systematic investigation and development of turbulence models for complex flows.

The objective of this paper is to review and identify the pertinent features that influence the selection of turbulence models, including shock/boundary-layer interaction, rotation, curvature, and separation. The paper includes a critical survey of various models available to predict such flows and their performance.

There have been several surveys published in recent years.[10-16] Reynolds[10] provided a classification of turbulence models, including a review of the attempts made to solve the entire set of constitutive equations governing turbulence. Rodi[11] and Launder et al.[12] reviewed the literature on turbulence models, with the main emphasis on the k-ϵ model; this review includes many of the important contributions made by the Imperial College and the Karlsruhe groups. The reviews by Marvin[13] and Wilcox and Rubesin[14] include critical examinations of the models used by NASA Ames groups for external flows, with an emphasis on the algebraic eddy viscosity and two-equation models, and cover a very comprehensive set of comparisons for various cases of compressible flows (including shock-induced flow separation). Hence, this aspect of turbulence modeling (external flows, for attached and separated flows) has been de-emphasized in the present paper. Lumley[15] and Launder[16] provide a futuristic view (higher-order closure models) of the scientifically based development of turbulence models. Most of

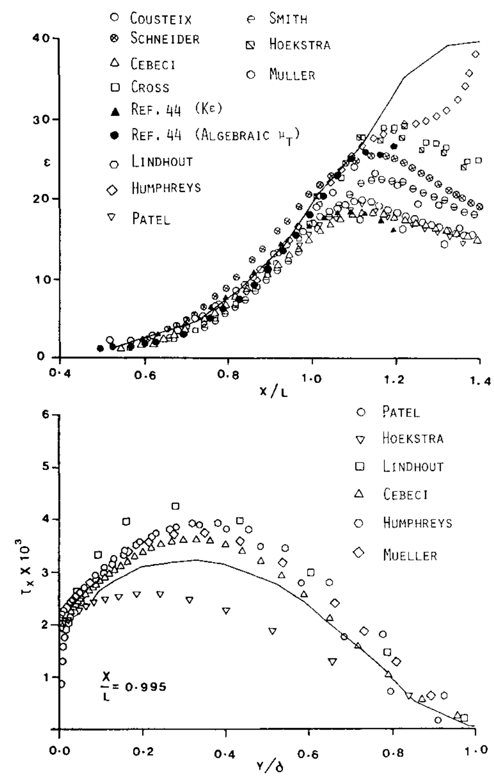


Fig. 1 Comparison between predictions[39] and three-dimensional boundary layer data on a finite wing[7] (ϵ is the limiting streamwise angle, L a characteristic length, x the distance from the leading edge, τ_x the principal stress coefficient).

the reviews described earlier are for "simple" shear and moderately complex flows. Furthermore, most of them are concerned with external flows, both two-dimensional and mildly three-dimensional.

No attempt is made in this review to duplicate the material from earlier surveys, with the exception of instances where it is presented in order to preserve the continuity to the development of complex models. This review should be considered as a complement to those that appeared earlier.[10-16] As a result of the many advances made in the modeling of complex flows in the last 5-10 years, the present review is timely.

Another objective of this review is to evaluate the various turbulence models proposed for complex flows, with an emphasis on curvature, rotation, and other effects. This paper covers the following topics: 1) three-dimensional flows, 2) flows subjected to longitudinal and transverse curvature, 3) flows subjected to rotation, 4) separated flows, 5) interacting boundary layers, and 6) flows with vorticity or vortex flows. To keep the review to a reasonable length, a decision was made not to include the following: 1) chemically reacting flows (e.g., combustion), 2) buoyancy effect, and 3) flows with heat-transfer effects. Furthermore, the compressibility effect is included in as far as the present objective is concerned, but no attempt is made to provide an exhaustive review in view of recent review papers on this topic.[13,14]

Equations for Incompressible, Rotating, and Curved Flows

Most complex flows have one or more extra features, as discussed above. It is advantageous to work with equations in generalized tensor form, since the direct effects of rotation, curvature, and other sources are included in the mean velocity and turbulence transport equations. Most of these equations can be found in the literature in Cartesian tensor form for a nonrotating case (e.g., see Ref. 17). The equations valid for the rotating case can be found in many papers and reports.[5,18-20] The equations are written for incompressible and Newtonian flow in a frame of reference rotating with steady angular velocity Ω. Buoyancy forces are neglected in this formulation.

The continuity equation is given by

$$U^i{}_{,i}=0 \quad u^i{}_{,i}=0 \tag{1}$$

The momentum equation is given by

$$(\rho U_i U^j)_{,j} = -2\epsilon_{ipj}\rho\Omega^P U^j - \rho[(\Omega_i x^j)\Omega_i - (\Omega_j \Omega^j)x_i] - (\bar{p}\delta_i^j + \overline{\rho u_i u^j} - F_i^j)_{,j} \tag{2}$$

where the first and second terms on the right-hand side are the Coriolis and centrifugal forces, respectively, and

$$F_{ik} = \text{viscous stress tensor } 2\mu \overline{S_{ik}}$$

$$\overline{F_{ij}u_{i,j}} = \rho\epsilon = \text{dissipation rate}$$

The Reynolds stress equation is given by

$$\underbrace{(\overline{\rho u_i u_k} U^j)_{,j}}_{1} = -(\underbrace{\overline{p' u_i}\delta_k^j + \overline{p' u_k}\delta_i^j}_{2a} + \underbrace{\overline{\rho u_i u_k u^j}}_{2b} \underbrace{- \overline{u_i F_k^j} - \overline{u_k F_i^j})_{,j}}_{2c}$$
$$+ \underbrace{(\overline{p' u_{i,k}} + \overline{p' u_{k,i}})}_{3} \underbrace{- (\overline{\rho u_i u^j} U_{k,j} + \overline{\rho u_k u^j} U_{i,j})}_{4}$$
$$\underbrace{- (\overline{u_{i,j} F_k^j} + \overline{u_{k,j} F_i^j})}_{5} \underbrace{- 2\rho\Omega^P(\epsilon_{ipj}\overline{u_k u^j} + \epsilon_{kpj}\overline{u_i u^j})}_{6} \tag{3}$$

The effect of rotation appears both explicitly and implicitly in the equation. The implicit effect or indirect effect is through its influence on various correlations. The curvature effects appear through the coordinate system employed in the generalized tensor form. The physical meaning of each of the terms in Eq. (3) is as follows. The first term represents the convection of Reynolds stresses and turbulence intensities ($i=j$) by the mean flow. The terms 2a, 2b, and 2c represent the diffusion of $\overline{u_i u_j}$ by the pressure gradient, turbulent fluctuations and viscous effects, respectively. The third term is the pressure-strain interaction. The fourth term is the production of $\overline{u_i u_j}$ by mean shear gradients. The fifth term is viscous dissipation of $\overline{u_i u_j}$ and the last term is generation/redistribution by Coriolis forces. It should be recognized that the effect of rotation is mainly in redistributing the intensities and shear stress.

The kinetic energy equation is given by

$$(\rho k U^j)_{,j} = -(\overline{u_i p'}\delta^{ij} + \overline{\rho k u^j} - \overline{u_i F^{ij}})_{,j} - \overline{\rho u^i u^j} U_{i,j} - \overline{F^{ij} u_{i,j}} \tag{4}$$

It is interesting to note that the rotation has no effect on the transport equation for kinetic energy and that its effect is mainly to redistribute the kinetic energy in three coordinate directions.

The equation for the dissipation rate in Cartesian tensor form is given in Ref. 17. Raj[20] derived the dissipation equation for rotating and curved flows, which in a simplified form is given by

$$\underbrace{(\rho\epsilon U^j)_{,j}}_{1} = -\underbrace{(\overline{\rho\epsilon u^j})_{,j}}_{2} \underbrace{-4\nu\overline{S^{ik}p'_{,ik}}}_{3} \underbrace{-4\mu\overline{S_{ik}u^j}\bar{S}^{i,k}_{,j}}_{4}$$
$$-4\mu\underbrace{[U^j_{,k}\overline{S^{ik}u_{i,j}} + U_{i,j}\overline{S^{ik}u^j_{,k}}]}_{5} + g^{nj}\mu\epsilon_{,nj} - \underbrace{4\mu\overline{S^{ik}u_{i,j}u^j_{,k}}}_{6}$$
$$\underbrace{-8\mu\epsilon_{ipq}\Omega^P\overline{S^{ik}u^j_{,k}}}_{7} \underbrace{-4\mu\nu g^{nj}\overline{S^{ik}_{,j}S_{ik,n}}}_{8} \tag{5}$$

In the above equation, the various terms represent, respectively, 1) convection by mean velocity, 2) diffusion by fluctuating velocity, 3) diffusion by fluctuating pressure gradients, 4) generation by mean flow, 5) viscous diffusion, 6) generation due to fluctuating velocity gradient, 7) generation/redistribution due to rotation, and 8) viscous destruction. It is interesting to note that the rate of dissipation is influenced by both the curvature and the rotation.

The equations of motion and turbulence transport for compressible flow is given by Rubesin[1] in Cartesian form. There are various methods of averaging the Navier-Stokes equation: conventional Reynolds averaging, "mass weighted averaging," and "volume averaging" are the most common. It should be remarked here that the influence of Mach number on turbulence models is significant only at Mach numbers in excess of $M=5$ (Ref. 21). The temperature and density fluctuations do not influence the turbulence field significantly up to at least $M=5$ and, hence, the turbulence models developed for incompressible flows are valid for compressible flows. This statement, known as Markovin's hypothesis, does not apply to free shear layers, where the turbulent fluctuations are much larger. Mixing layers exhibit Mach number effects for $M>1.5$. The reacting flows, buoyant flows, and flows with significant heat transfer (as in combustion) need special attention. In such flows, the equations of motion and energy for compressible flow, along with the incompressible turbulence models, are to be employed.

Structure of Turbulence in Three-Dimensional, Rotating, Curved Flows

Before we provide a review of various turbulence models for complex flows, it is first important to understand the physical phenomena associated with these effects. Some representative cases will be covered to illustrate the "extra strain" effect on complex flows. This provides the necessary foundation for understanding the turbulence models used for these complex flows. A brief discussion on physical aspects of various complex shear flow is given in Ref. 8.

One of the complex flows is that with streamline curvature. Bradshaw[5] provided an excellent review of the literature on this topic. It is well known that the centrifugal force supresses the turbulence on a convex surface and amplifies the production on a concave surface. It also depends on the angular momentum gradient. A wall jet, whose angular momentum decreases with radius, is unstable on a convex surface. Such effects are evident from boundary-layer measurements taken on curved surfaces, curved wall jets, swirling turbulent flows, and in trailing vortices.

Three-dimensional boundary layers are very common in practice. The presence of cross flows and the additional strain due to the presence of $\partial W/\partial y$, $\partial U/\partial z$, and other velocity gradients would alter the structure of turbulence, length scale, intensities, etc. The direction and magnitude of shear stress differ considerably from those calculated/inferred from two-dimensional concepts. The three-dimensional boundary-layer data reviewed[22-24] clearly reveal this effect. But the two major effects observed, which indicated the complexity of such flows, are the following:

1) The eddy viscosities ($\tau/\partial U/\partial y$) deduced from the data deviates considerably from those calculated from two-dimensional values.

2) The direction of the resultant stress and resultant velocity gradient deviate considerably. However, even in a two-dimensional flow, it is well known that the principal stress axes and principal strain axes are very different.

The analysis of data by Johnston[23] show the dramatic effect of three-dimensionality. The ratio of local streamwise to local cross-flow eddy viscosity shows a large variation (0.1-1.0) in the three-dimensional boundary-layer data from various sources.

The effect of rotation on turbulence structure was reported in Refs. 2 and 25. Anand and Lakshminarayana investigated the turbulence properties in an axial turbomachinery channel, while Johnston and his group[3,25] investigated the flow in a modeled centrifugal compressor. The model used by Johnston's group was a constant-area channel rotating about the spanwise axis. On the basis of mixing layer concept, they proved that the Coriolis force has a destabilizing effect (amplification) on the leading side and stabilizing effect (attenuation) on the trailing side of the rotating channel. The results in Ref. 2 indicate that the effect of Coriolis force is to redistribute turbulent kinetic energy inside the boundary layer, with an increase in turbulence intensities in the radial direction (direction of the Coriolos forces) and a corresponding decrease of turbulence quantities in the streamwise direction.

Separated flows present one of the most complex flows, in terms of both physical understanding and modeling. Simpson's[26] group has carried out a systematic study of these flows. He argues, based on various data available, that the small-scale mean back flow does not come from downstream flow, but from large-scale coherent eddies passing through the separated region. The inner layer in a separated flow is substantially different, phenomenologically, from that in an unseparated flow layer. Large turbulence fluctuations occur in this region, the energy for the flow mainly coming from the large-scale structures. Simpson suggests "Reynolds stresses in this region must be modeled by relating them to the turbulence structure and not to the local mean velocity gradients." This suggests major limitations of the present-day models based on the concepts derived from the simple shear layer. The other feature that is distinctly different from the simple unseparated shear layer is that the normal stresses ($\overline{v^2}$, $\overline{uw}$) has a dominant effect on the mean and turbulence equations.

Modeling Summary

A general classification of various methods and levels of closure is given by Ferziger et al. in Ref. 1. The models can be generally classified as eddy viscosity or Reynolds stress models. There are various levels of eddy viscosity models, the simplest being the Boussinesq.[27] In this concept, the turbulent stresses in the mean momentum equation ($\rho\overline{u_i u_j}$) are expressed as

$$\rho\overline{u_i u_j} = \nu_t (U_{i,j} + U_{j,i}) \tag{6}$$

where the constant ν_t, called the "eddy" viscosity, is proportional to the mean velocity field U_i. The physical idea behind this concept is explained in Ref. 17. The stresses in laminar flows arise due to random molecular motion, which is similar to turbulence fluctuation. The resemblance between these two motions is somewhat superficial. Nevertheless, it is assumed that the transfer of momentum and heat by molecular motion is similar to that caused by the turbulence fluctuation. The concept of "eddy viscosity" is phenomenological and has no mathematical basis. It should be emphasized that the molecular viscosity is a property of the *fluid* and that turbulence is a property of the *flow*. Hence, the eddy viscosity is likely to be a function of the flow properties (e.g., mean velocity) and it may also be a vectorial quantity in three-dimensional flow.

The eddy viscosity can be prescribed algebraically in terms of quantities derived (e.g., k, ϵ, or ω) from partial differential equations. Depending on the type and number of equations employed, they are called algebraic, one-, and two-equation models.

Classification and review of the turbulence models (as of 1976 or earlier) are given in Ref. 10. The closure equation can be employed in combination with the time-averaged momentum, continuity, and energy equations. The methods can be classified as follows:

1) Zero-equation or algebraic eddy viscosity model. These models employ an algebraic form for the eddy viscosity v_t.

2) One-equation model. This model employs an additional partial differential equation (PDE) relating to the turbulence velocity scale. ($\nu_t = VL$, where L is the character length scale and V the characteristic velocity scale.)

3) Two-equation model. This model employs one PDE for relating the turbulence length scale and one for relating the turbulence velocity scale. [$\nu_t = C_\mu k^2/\epsilon$ or $\nu_t = \gamma^* k/\omega$, where the transport equations (PDE) for k and ϵ, or k and ω are employed.]

4) Modified two-equation models (MTEM). In many flows, predictions with a constant value of C_μ are found to be inadequate. Many authors have tried to modify the constant to bring in more physics. Some of them are arbitrary and others are based on a simplified (algebraic) Reynolds stress equation.

5) Algebraic Reynolds stress model (ARSM). In this model, approximations are used to reduce the PDEs governing the Reynolds stresses to an algebraic form. These equations are coupled to the transport equations of the two-equation model (item 3 above) to derive the turbulence and flowfield.

6) Reynolds stress model (RSM). This model employs several PDEs for the components of the turbulence stress tensor ($\overline{u_i u_j}$). This is one of the most complex models in use today.

7) Large-eddy simulation. The time-dependent (three-dimensional) large-eddy structure is resolved through a numerical solution of time-dependent Navier-Stokes equations and a low-level model for the small-scale turbulence is employed.

One can group the zero, one-, and two-equation models as eddy viscosity models and the MTEM and ARSM as pseudo-eddy-viscosity models, as the C_μ is no longer a constant, but

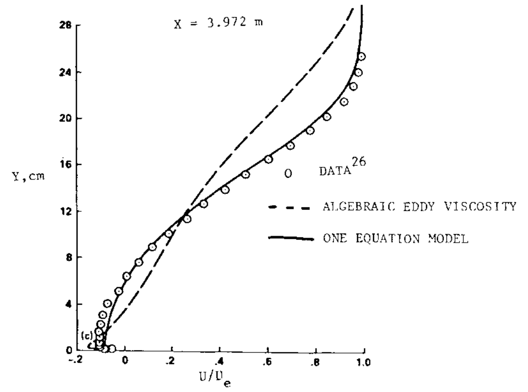


Fig. 2 Low speed diffuser flow; mean velocity profile comparisons[46] (X is distance from entrance, Y distance normal to wall).

rather a function of the local turbulence and mean flow properties.

The zero-equation or algebraic eddy viscosity models are widely used in practical engineering applications involving simple shear flows and in many of the Navier-Stokes computer codes. The one-equation model was widely employed in early stages of turbulence model development and are still in use in limited regions such as the near-wall sublayer. In these regions, the one-equation model is much simpler to use than the more complicated models. The two-equation model is employed when additional details on turbulence quantities are needed. The Reynolds stress equation is under extensive development and is employed in complex flow situations, e.g., three-dimensional flows, flow with curvature and rotation, and blowing and suction. This model is essential if details of the turbulence as well as accurate flow prediction are needed. The numerical computation involving large-eddy simulation is prohibitively expensive and its use is presently limited to very simple flows (flat-plate boundary layer, channel flows, etc.).

Algebraic Eddy Viscosity Model

The eddy viscosity formulation is based on the law of the wall or mixing length concept. The "mixing length model" can be written as

$$\mu_t = 2\rho\ell^2\sqrt{S_{ij}S_{ij}}, \qquad S_{ij} = (U_{i,j} + U_{j,i})/2 - \tfrac{1}{3}\delta_{ij}U_{k,k} \qquad (7)$$

It is well known that the eddy viscosity is not constant across a boundary layer. Furthermore, the turbulence is suppressed close to the wall; hence, a correction is applied to the above equation to satisfy the wall condition and allow for its variation across the boundary layer. Several groups have made extensive modifications to the algebraic eddy viscosity model and have used it for the computation of turbulent flowfields.[28,29]

One of the most widely used models for the engineering calculation of boundary layers is that due to Cebeci et al.,[28] later modified by Baldwin and Lomax.[29] The model proposed by Baldwin and Lomax has wider applications and avoids the necessity of finding the edge of the boundary layer. A critical examination of this model, its application, and a comparison with various data are given in Ref. 13 and will not be repeated here.

The model assumes the flow to be isotropic and is strictly valid for two-dimensional "simple shear flows." Various data and comparisons available (e.g., Refs. 1, 9, 13, 14, and 28-34) indicate the following conclusions:

1) The model is adequate for two-dimensional compressible flows with mild pressure gradients. The mean velocity field is predicted well, but the stress prediction is only qualitative.

2) The model is suitable for three-dimensional boundary layers with small cross flows, but may not be suitable for flows with large cross flows. When memory effects are present, the algebraic viscosity model provides poor predictions.

3) The model is not suitable for flows with curvature, rotation, and separation. The model is of little value in three-dimensional complex flows or in situations where turbulence transport effects are important.

4) Assumption of isotropy is not valid for pressure- and turbulence-driven secondary flows or when abrupt changes in the strain or shear rate are present.

5) The model cannot predict shock-induced separated flow, as it fails to capture the turbulence amplification that occurs across a shock.[35] The reattachment or downstream flow is not predicted accurately. The model fails to capture the full rise in the reattachment region. Relaxation models have not been successful. Many of the constants used in the model are functions of Mach number.

The recommendations are:

1) The use of the model should be restricted to two-dimensional flows with mild pressure gradients and mild curvatures with no flow separation and/or rotation effects.

2) It is useful for two-dimensional shock-separated boundary layers with weak shocks, as in the case of external flows.

3) It is useful in computing three-dimensional boundary layers with small cross flows and mild pressure gradients having no curvature or rotation effects.

It should be noted that for an attached two-dimensional boundary layer with a mild pressure gradient, there is very little advantage in resorting to a higher-order model. The algebraic eddy viscosity model is adequate for the prediction of the mean velocity field.

There have been several attempts to improve the algebraic eddy viscosity model to account for the extra strain, but these attempts have had limited success. For example, based on Bradshaw's work,[36] a correction is applied for the length scale ($\nu_T = VL$) and the correlation is given by

$$L/L_0 = 1 - \beta R_i \qquad (8)$$

where β is an empirical constant and R_i the gradient Richardson number. For curved flows, this parameter is given by

$$\frac{L}{L_0} = 1 + \beta \frac{e}{\partial U/\partial y}$$

where L_0 is the length scale of the simple shear layer and e the extra strain rate. For example, $e = 2U/R$ for flows with longitudinal curvature R and $e = 2\Omega$ for flow systems with rotation. A detailed discussion of the applicability of this equation is given in Ref. 5. The value of β is nearly equal to seven for a convex surface ($R_i > 0$, stable conditions) and four for a concave surface ($R_i < 0$, unstable conditions). But recent measurements[37] indicate that the value of β varies across the boundary layer. The inner (linear) region is well represented by this equation and $\beta = 7$, but for $0.2 < y/\delta < 0.6$, $\beta = 10$ provides better results; beyond this region the agreement is poor. Castro's (see Bradshaw in Ref. 1) data indicate that this value is closer to 2.1 for a curved mixing layer.

Johnston's[25] experiment on a rotating channel shows that values of $\beta = 4$–6 may be acceptable in this flow. The value of coefficient β (Monin-Oboukhov) was found[4] to depend on the rotation number based on the distance from the inlet of the test section ($\Omega x/U_e$). This clearly reveals the controversy associated with this formulation. It is not clear whether a simple relationship such as Eq. (8) will provide a predictive technique. It may be suitable for mild curved flows and mild rotation, where the extra strain effects are small.

It should be emphasized that very few researchers have reported attempts to predict complex flows using algebraic eddy viscosity and the failures are very rarely reported in the open literature. The NASA Ames Group[13] has made extensive use of the algebraic turbulence model to predict some of the complex flows and found the results to be inferior to those based on higher-order models.

An extension of the algebraic eddy viscosity model to three-dimensional flow has been carried out by Rotta.[38] He writes

$$-\overline{uv} = \nu_t\left(a_{xx}\frac{\partial u}{\partial y} + a_{xz}\frac{\partial w}{\partial y}\right)$$

$$-\overline{vw} = \nu_t\left(a_{xz}\frac{\partial u}{\partial y} + a_{zz}\frac{\partial w}{\partial y}\right) \tag{9}$$

where $a_{xx} = 1-(1-T)\sin^2\beta$, $a_{xz} = (1-T)\sin\beta\cos\beta$, $a_{zz} = 1-(1-T)\cos^2\beta$, and T is a prescribed constant, and $\tan\beta = W/U$. This model has been used by Cousteix and Mueller (both papers are in Ref. 22) for three-dimensional shear flow. The model has not been very successful; the value of T had to be varied from one flow to another for good prediction.

Nakkasyan and Rhyming (see Refs. 22 and 39) have introduced an anisotropic eddy viscosity model for three-dimensional flow, given by

$$-\overline{uv} = \nu_t E_s\left(\frac{\partial U}{\partial y}\right) \qquad -\overline{vw} = \nu_t E_n\left(\frac{\partial W}{\partial y}\right)$$

where E_s and E_n are functions of mean velocity shear stresses and curvature of the streamline.

In a recent workshop held in Berlin, following a symposium on three-dimensional turbulent boundary layers,[22] an attempt was made to evaluate the predictive techniques for three-dimensional boundary layers, including turbulence modeling. The three-dimensional shear layer data from Refs. 7 and 40–43 were used in the evaluation process. Most entries in the workshop utilized algebraic viscosity models or modifications thereof. There was one computation based on the k-ϵ model and one that utilized the Reynolds stress model. The comparison of the data from DeChow and Felsch with the predictions is given in Refs. 8 and 39. Most of these use the boundary-layer solution technique (Crank-Nicholson, Keller box, zig-zag method, etc.) and all of them utilize the algebraic viscosity model, with small variations. It is clear that only Hoekstra[39] predicts the mean velocity reasonably well and none of the predictions for shear stress are good.

On the other hand, most of the computors predicted the streamwise velocity profiles measured on a swept wing by Van den Berg et al.,[7] as well as the limiting streamline angle (deviation of the streamline angle from the freestream as the wall is approached) in the small cross-flow region (see Fig. 1). But none of the predictions are good for the large cross-flow region nor at locations near and beyond the separation point. Cebeci's prediction of the streamwise stress τ_x is good, but the predictions from others depart considerably from the measured data. The recent computations by Murthy and Lakshminarayana,[44] shown in Fig. 1, indicate that the algebraic eddy viscosity model is not adequate and that predictions from the two-equation model are slightly better. But even this latter model is not adequate for predictions near the separation point.

One-Equation Model

A modeled equation for the turbulent kinetic energy [Eq. (4)] is used in the formulation of the one-equation model. The detailed development and review of this model are given in Refs. 10 and 45. The velocity scale is derived from the turbulent energy equation and the length scale is derived from a model similar to the algebraic eddy viscosity model ($\nu_t = VL$). It is now recognized that the characterization of turbulence by one transport equation for V is not adequate.

There has been a recent attempt to revive this model for complex flows. One such attempt is by Johnson and King[46] to predict the two-dimensional separated flows. The turbulent kinetic energy [similar to Eq. (4)] is used to describe the development of the Reynolds stress (assuming that the ratio of the local kinetic energy to the shear stress is constant) in conjunction with an assumed eddy viscosity distribution, which has as its velocity scale the maximum Reynolds stress. Various assumptions are made to reduce the PDE for k to a one-dimensional equation for Reynolds stress. Hence, this model would fall between the zero- and the one-equation models. As the authors have pointed out, their approach is to provide a model for a limited class of flows (two-dimensional separated flows without curvature and rotation effects) rather than a universal model.

The prediction of two-dimensional separated flows in a diffuser, shown in Fig. 2, indicates that their prediction of mean velocity profile in the separated region is better than the predictions from the algebraic eddy viscosity model. The NASA Ames group has had considerable success in computing many high-speed flows and transonic shock-separated flows with this model.

Two-Equation Models

The two-equation model employs more physics than either the algebraic viscosity model or the one-equation model. There are two basic models available. One of these was developed by Jones and Launder[47] and has been modified and used to predict a large class of two-dimensional and mildly complex flows. The model by Saffman[48] and Wilcox and his group[14,49] has been employed to predict many flows. Details on the derivations of these model equations can be found in the references quoted above and in Refs. 10 and 12.

The two-equation models can be classified as follows.

Kinetic Energy-Dissipation Equation (k-ϵ) Model

In this model the eddy viscosity is represented by

$$\mu_t = C_\mu \rho \frac{k^2}{\epsilon} \tag{10}$$

where the length scale $\ell = k^{3/2}/\epsilon$ and velocity scale is $\sqrt{k}$. The constant C_μ is scalar for isotropic turbulence. For non-isotropic flows (e.g., three-dimensional rotation flows), C_μ should be a vectorial quantity and is thus no longer a constant. This model employs transport equations for k and ϵ. Jones and Launder[47] modeled various terms in the kinetic energy equation and the dissipation equation to provide the following simplified turbulence transport models. Their equations were generalized and slightly modified by Galmes and Lakshminarayana[50,51] to include the effect of rotation and the curvature. These equations are given by

$$(\rho k U^j)_{,j} = \left[\left(\mu + \frac{\mu_t}{C_k}\right) g^{ij} k_{,i}\right]_{,j} + P - \rho\epsilon - \frac{10}{3} g^{ij}\mu k^{1/2}_{,i} k^{1/2}_{,j} \tag{11}$$

where $P = -\rho\overline{u_i u^j} U_{i,j}$ is the production

$$(\rho\epsilon U^j)_{,j} = \left[\left(\mu + \frac{\mu_t}{C_\epsilon}\right) g^{ij}\epsilon_{,i}\right]_{,j} + C_{\epsilon 1} F_\epsilon (R_T, R_{ic}) \frac{\epsilon}{k} P$$

$$- C_{\epsilon 2}\rho\frac{\epsilon^2}{k} + C_{\epsilon 3}\mu\frac{k^2}{\epsilon} g^{ij}(\bar{S}_{\ell k,i}\bar{S}^{\ell k}_{,j}) \tag{12}$$

where

$$C_\epsilon = 0.23, \quad C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.92, \quad C_{\epsilon 3} = 0.2$$

and

$$F_\epsilon(R_T,R_{ic}) = 1 + 0.3(1-R_{ic})\exp(-R_T^2) \tag{13}$$

These equations are identical to those of Launder and Jones, with the following exceptions: 1) the constant in the last term of Eq. (11) is zero for high Reynolds number flows and away from the wall and in situations where the wall functions are used; 2) the function F_ϵ in Eq. (12) includes the effect of rotation; and 3) the last term in Eq. (12), which is due to the wall effect, is negligible for most high Reynolds number flows. This term is a generalization of the term proposed by Launder and Jones.

Galmes et al.[51] computed the flow over a rotating cylinder[42] and found that the last terms in Eqs. (11) and (12) and F_ϵ had no significant effect on the predicted values. But, there may be other situations in which these terms are important.

The standard k-ϵ model [without the last two terms in Eqs. (11) and (12), and $F_\epsilon = 1$] has been used, with and without minor modifications, to predict a wide variety of flows: jets, wakes, boundary layers, separated flows, etc. (see e.g., Refs. 1, 9, 10–13, 14, etc.).

In many complex flows, the flowfield near the wall has to be determined accurately. The use of wall functions is not satisfactory. Chien[52] proposed modifications to the k-ϵ equation near the wall. Chien provided an analysis of the near-wall effect and used the term $(2\nu k/y^2)$ instead of the last term in Eq. (11) and, in place of the last two terms in Eq. (12), he introduced the term

$$-\frac{\rho\epsilon}{k}\left[C_2 f\epsilon + \frac{2\nu k e^{-(C_4 u_* y/\nu)}}{y^2}\right] \tag{14}$$

where $f = 1 - 0.22\exp(-k^2/6\nu\epsilon)^2$.

The prediction of the flat boundary layer shows very good agreement near the wall for a flat-plate boundary layer. The NASA Ames group has found deficiencies in this model near separation and reattachment. In a recent paper, Bernard[53] reports that all approaches fail to account for the large peak values in k in the wall region and attributes this defect to a fundamental inconsistency in the commonly used pressure diffusion term in the k equation near the boundary.

Kinetic Energy-Dissipation Rate Model (k-ω)

In this model, the eddy viscosity is expressed as

$$\mu_t = \gamma^* \frac{k}{\omega} \tag{15}$$

where ω is the specific dissipation rate. This model is due to Soffman and his group[14,48,49] and requires transport equations for k and ω, given by

$$\frac{D}{Dt}(\rho k) = \rho\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^*\rho\omega k + \frac{\partial}{\partial x_j}\left[(\mu+\sigma^*\rho\epsilon)\frac{\partial k}{\partial x_j}\right] \tag{16}$$

$$\frac{D}{Dt}(\rho\omega^2) = \gamma\frac{\omega^2}{k}\rho\tau_{ij}\frac{\partial u_i}{\partial x_j} - \left[\beta + 2\sigma\left(\frac{\partial \ell}{\partial x_k}\right)^2\right]\rho\omega^3 + \frac{\partial}{\partial x_j}\left[(\mu+\sigma\rho\epsilon)\frac{\partial\omega^2}{\partial x_j}\right] \tag{17}$$

where

$$\ell = k^{1/2}/\omega; \quad \beta = 0.06; \quad \beta^* = 0.09; \quad \sigma = \sigma^* = 1/2$$

$$\gamma^* = [1-(1-\lambda^2)\exp(-Re_T/Re)]$$

$$\gamma\gamma^* = \gamma_\infty[1-(1-\lambda^2)\exp(-Re_T/R_\omega)]$$

$$\gamma_\infty = \frac{10}{9}, \quad \gamma = \frac{1}{11}, \quad Re = 1, \quad R_\omega = 2, \quad Re_T = \frac{k^{1/2}\ell}{\nu}$$

Detailed predictions from this model can be found in Refs. 13, 14, 48, and 49. This model has been used by a limited number of researchers and, hence, no judgment can be made with regard to its performance compared to the k-ϵ model.

The two-equation models, without modifications, fail to capture many of the features associated with complex flows. The prediction of the flows with curvature, separation, vortex, and rotation are not good with these models; but they are adequate for most simple flows without extra strain. For example, Barton and Birch[54] showed that the mean flow and turbulence quantities in a seven-lobe mixer (the flow is predominantly axial, with no swirl) are predicted accurately with a k-ϵ model. But, the model fails in complex situations. It is noted in Refs. 55 and 56 that the two-equation models do not predict the inception of separation or the reattachment accurately. Consequently, the predictions of C_p, C_f, and velocity profiles are not predicted accurately in and around the separated regions. A comparison of the separation point and reattachment point, measured and predicted, over a bump in an axisymmetric cylinder[56,57] clearly shows the inability of the k-ϵ model to capture such flows.

Gorski et al.[58] carried out a systematic study on the effects of "slip" and "no-slip" conditions for the wall on computed results, using both the algebraic eddy viscosity and Jones and Launder's k-ϵ model. The flow in a wing/body junction was computed.[59] When a no-slip boundary condition was used, the k-ϵ model was found to be significantly better than the algebraic eddy viscosity model (Fig. 3). The predictions for both models were found to be almost identical when a slip condition based on the law of the wall is employed. It should be mentioned here that, even though the flow is three-dimensional in this case, the cross flows and transverse velocities are small; hence, the flow is very mildly three-dimensional.

The advantages and simplicity of the k-ϵ model (as compared to the Reynolds stress model) should not be overlooked. It is much superior to the algebraic eddy viscosity or one-equation model in mildly complex flows. A good example of this is the prediction of viscous flow through a cascade carried out by Kirtley and Lakshminarayana.[31] A space-marching technique of solving the Navier-Stokes equation was utilized to predict the boundary-layer growth on a cascade of blades. The predictions from the k-ϵ model are excellent and, in fact, are better than those of the algebraic eddy viscosity model in the near wall region.

Hah and Lakshminarayana[60,61] have used the standard k-ϵ model to compute the flowfield in a compressor rotor wake and found that the mean velocities can be predicted reasonably well with a k-ϵ model; however, the predictions of turbulence quantities are improved substantially with an algebraic Reynolds stress model (ARSM), described below. When the curvature is present, they found that the predictions are improved with the ARSM.

Recent investigations at the Pennsylvania State University[50,51, 60-62] indicate that in situations where significant rotation and curvature effects are present, the k-ϵ model alone does not perform as well as the ARSM in predicting these flows. A detailed comparison will be provided below.

Some of the conclusions concerning the k-ϵ model (without modifications), based on the results extracted from various references on k-ϵ model computation are as follows:

1) The k-ϵ equations have been widely used for two-dimensional flows with pressure gradients. Mean velocities are predicted fairly accurately for these cases and the gross properties of turbulence are predicted well. The model is good for attached two-dimensional boundary layers. For these cases, the Reynolds stress model has little advantage over the k-ϵ/k-ω models.

2) Predictions are good for two-dimensional flows in jets, channels, diffusers, and annulus wall boundary layers without swirl or separation.

3) The model is inadequate for separated flows and abrupt changes in the strain rate or shear. The performance of k-ϵ/k-ω models is good for mildly separated flows and recirculating flows.

4) The predictions are not good for three-dimensional flows. The shortcomings are due to the models used for the pressure strain term, the assumption of isotropy, and the low Reynolds number formulation in the near-wall vicinity. Isotropic eddy viscosity is adequate for three-dimensional boundary layers with very small cross flow, but fails for significant cross flows and swirl.

5) The model fails for flows with rotation, curvature, strong swirling flows, three-dimensional flows, shock-induced separation, etc. When curvature is present, the models including curvature effects perform much better.

6) Most comparisons between the model and data are made for standard and simple experimental data.

7) For two-dimensional supersonic flow with a large adverse pressure gradient, improved predictions in regions of rising C_f are observed with the k-ϵ model as compared to the algebraic eddy viscosity models.

The following recommendations are made with regard to the k-ϵ/k-ω equations:

1) The model is adequate for two-dimensional flows with pressure gradients, two-dimensional recirculating flows, two-dimensional jets, three-dimensional flows with very mild cross flows, and unseparated two-dimensional boundary layers.

2) The model needs modification to include anisotropy, curvature, and rotation effects on "constants," turbulence amplification through the shock wave, and description of stress components for three-dimensional flows. The range of applicability of these models can be extended by coupling k-ϵ equations with algebraic stress equations ($\overline{u_i u_j}$) to account for nonisotropic effects.

3) Several possible reasons for some of the failures are that the constants used in these models are based on well-documented two-dimensional flows and the assumption of isotropy, and that only the gradient-induced diffusion is included in the model.

Several attempts have been made to modify the k-ϵ/k-ω equations to extend the range of validity of these models to complex flow situations. A brief summary of these attempts is given below. As mentioned earlier, no attempt has been made in this review to include the effect of density fluctuations, which is important in some supersonic flows and reacting flows. Various models available for including these effects, which are important in environmental and combustion problems, are reviewed by Rodi.[63]

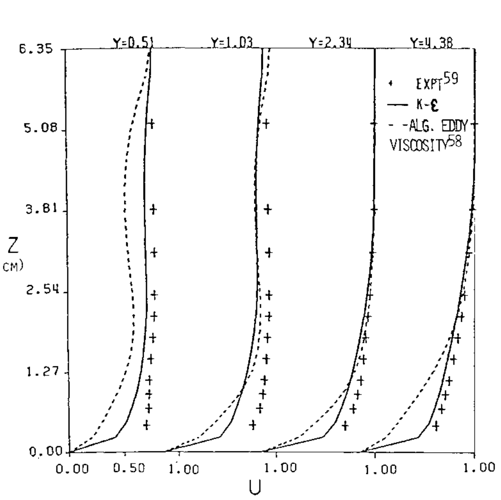


Fig. 3 Comparison of streamwise velocity computed using k-ϵ and eddy viscosity models with no-slip boundary condition at $x = 0.647$ m on a wing/body junction (U is normalized by U_e, Z the distance normal to wall, Y the distance to wing).

When the curvature, rotation, and other extra strain effects are present, the Reynolds stress closure equations can provide a more realistic and rigorous approach to account for these complex strain fields. But these are prohibitive from a computational standpoint. Hence, simpler approaches based on the two-equation models have been pursued. These models require empirical modeling of C_μ or modification of the constants in the k and ϵ transport equation. Some of these attempts are described below.

Modifications of the k-ϵ/k-ω Model for Curvature and Rotation Effects

Several investigators (e.g., Refs. 49, 50, 51, 60–67) have attempted to modify the two-equation models to include the effects of curvature and rotation. Most of these attempts utilize an algebraic version of the Reynolds stress equation to derive an expression for C_μ. In such a case, C_μ is a function of the local values of the P, ϵ, and velocity gradients. Several other researchers have solved the k-ϵ/k-ω equations coupled with the algebraic Reynolds stress equation. These attempts will be described in a later section. Only those attempts that modify the k-ϵ/k-ω equation without resorting to the ARSM are described in this section.

As indicated earlier, the energy equation (11) is not affected by rotation or curvature. The direct effect (as opposed to the indirect effect arising through modeling) is included in Eq. (11).

Launder et al.[65] modified one of the empirical coefficients in Eq. (12), whose magnitude is directly proportional to the Richardson number based on a time scale of energy-containing eddies. The modified coefficient $C_{\epsilon 2}$ in Eq. (12) is given by

$$-C_{\epsilon 2}(1 - C_c R_{it}) \tag{18}$$

where R_{it} is a turbulent Richardson number that depends on k, ϵ, swirl velocity, and curvature. The value of $C_c = 0.2$ provided reasonably good predictions for the flow over a rotating cone and for the boundary-layer data on a curved surface.

Most of the corrections to rotation are done on the dissipation term in Eq. (12).[64,65] The analysis of Refs. 50, 51, and 65 indicates that the rotation should affect the production term instead of the dissipation term, as implied in Eq. (12).

There have been several attempts to incorporate the Coriolis force effect in k-ϵ/k-ω models. Wilcox and Chambers introduced a term (see Ref. 49 for physical reasoning), $9\Omega\mu_t(\partial U/\partial y)$ in the k equation. Howard et al.[64] used this model and proposed an additional model based on Ref. 65, suggesting that $C_{\epsilon 2}$ in Eq. (12) be replaced by

$$C_{\epsilon 2}\left[1 + 0.2\left(\frac{k}{\epsilon}\right)^2 2\Omega\left(\frac{\partial U}{\partial y} - 2\Omega\right)\right] \tag{19}$$

The results from both of these modifications[49,64] are compared with the Johnston et al.'s[3,18] rotating duct data in Figs. 4 and 5. The predictions, even though they show the correct trend, are at best qualitative. Furthermore, it is evident from the discussion that such arbitrary modeling and introduction of the rotation effect through "intuitive" reasoning is not adequate. Hence, the reviewer feels that such attempts should be abandoned in favor of more rigourous modeling.

Modeling for Reynolds Stress Transport Equations

As indicated earlier, neither the algebraic eddy viscosity nor the two-equation models (standard) can provide an adequate prediction of the complex flowfield. The most logical choice would be to introduce additional physics/constitutive equations that govern the structural changes and extra strain effects through the use of the Reynolds stress transport equation. The Reynolds stress equation in the form given in Eq. (3) is intractable.

Before we proceed to review turbulence models utilizing the Reynolds stress equation in one form or the other, it is essential to briefly summarize the modeling effort. The Reynolds stress equation (3) contains many terms that need to be modeled. A brief review of these efforts are given in Refs. 10-17 and 68. Some of the previous work on Reynolds stress modeling was modified to include the rotation terms[50,51]; wherever possible, the generalized form is included.

There have been numerous attempts to model the Reynolds stress equation, to simplify it, and to apply it to some specific complex flows. The terms that need to be modeled in Eq. (3) are: diffusion (term 2), pressure strain correlation (term 3), and viscous dissipation (term 5). These have to be expressed in terms of mean velocities, Reynolds stresses, and their derivatives, including the rotation and curvature effects.

Excellent reviews by Lumley,[68] Rodi,[1] and Launder et al.[12] provide details of the lower- as well as the higher-order modeling. The modeling of the pressure-strain correlation [term 3 in Eq. (3)] has been pursued by many (e.g., Refs. 12, 50, 68-74). Based on these earlier models, the following model was proposed for the generalized case with rotation[50,51] (in Cartesian tensor form):

$$\frac{\overline{p'u_{i,k}+p'u_{k,i}}}{\rho}=\left[-C_1\frac{\epsilon}{k}\left(\overline{u_iu_k}-\frac{2}{3}\delta_{ik}k\right)-C_2\left(P^*_{ik}-\frac{2}{3}\delta_{ik}P\right)\right]F\left(\frac{\ell}{n_ix^i}\right) \tag{20}$$

where

$$P^*_{ik}=-\overline{u_iu_j}U^*_{k,j}-\overline{u_ku_j}U^*_{i,j}$$

$$U^*_{k,j}=U_{k,j}+\epsilon_{kpj}\Omega^P$$

C_1 and C_2 are modeling constants, and n_i a unit vector.

The pressure fluctuation is expressed in the form of a volume integral,[69] which shows that there are three kinds of interactions: one involving fluctuating quantities; another arising from the presence of external effects, such as the mean strain rate and the rotation; and a last term involving interactions with a solid boundary, which is small for flows away from the solid boundary. Inside the first set of parentheses on the right side of Eq. (20) is what is called the "return to isotropy term," which acts to interchange the energy among the components when the turbulence is anisotropic and which vanishes when the turbulence is isotropic. The terms in the second set of parentheses on the right of Eq. (20) are called the "rapid terms."

The function F in Eq. (20) represents the wall effect and is given by

$$F\left(\frac{\ell}{n_ix^i}\right)=1+2.5\frac{k^{3/2}}{\epsilon n_ix^i} \tag{21}$$

Bertoglio[75] carried out a theoretical study of the effect of rotation on the structure of homogeneous turbulence, especially the pressure-strain correlation. The results indicate that the "rapid part" of the pressure strain is affected by the rotation. The direction as well as the magnitude of the Coriolis force affect the structure, so the isotropic assumption is not adequate. The effect of rotation and curvature on the pressure-strain correlation is not properly understood. There is a need for a systematic experimental and analytical study of physically realistic flows to model the pressure-strain term adequately to account for extra strain arising from rotation, curvature, three-dimensionality, etc.

The dissipation term [term 5 in Eq. (3)] is modeled by assuming that the dissipative motions are isotropic.[70] Several experimental studies have shown that the turbulence does not remain locally isotropic in the presence of strong strain fields.[76] The following form proposed by Hanjalic and Launder[77] seems to be more general and is probably adequate for most flow:

$$D_{ik}=-\frac{2}{3}\rho\epsilon^*\left(\delta_{ik}+\frac{k}{\epsilon}R_T^{-1/2}\bar{S}_{ik}\right) \tag{22}$$

where

$$\epsilon^*=\epsilon+\frac{10}{3}g^{\ell m}\nu k^{1/2}_{,\ell}k^{1/2}_{,m}$$

Launder and Reynolds[78] have recently shown that this is not an exact limiting form and have proposed addition of several terms to this equation.

For high Reynolds number flows, the last term is negligible and the dissipation term reduces to

$$D=-2/3\rho\epsilon\delta_{ik} \tag{23}$$

The above expression assumes that the dissipative motion is isotropic. A discussion of these and other advanced models can be found in Refs. 68, 73, and 79. The modeling of viscous dissipation terms is still not in a satisfactory state. Many researchers have proposed using a transport equation for ϵ dissipation [Eq. (5)]. This is one of the pacing items in turbulence modeling and we may see the emergence of sophisticated models at a later date. Even if these modeled transport equations for dissipation become available, it will be for simple flows. It is clear from Eq. (5) and discussions in Refs. 20, 50, and 51 that the rotation and curvature effects are important. Hence, for the present and near future, the engineering calculations have to proceed with Reynolds stress closure and modeling given by Eqs. 20-23.

With regard to these diffusion terms, Launder et al.[70] neglected the diffusion due to pressure fluctuations and modeled the triple velocity correlation by

$$\overline{u_iu_ju_k}=-C_s\frac{k}{\epsilon}\overline{u_ku_m}\frac{\partial\overline{u_iu_j}}{\partial x_m} \tag{24}$$

Hence, Eq. (3), with models represented by Eqs. (20), (22), or (23), and (24), represents the Reynolds stress closure equation. A discussion of the effects of rotation on the Reynolds stress equation can found in Refs. 2, 3, 20, 25, 60, 74, 75, 80-83. A similar discussion on the effects of curvature can be found in Refs. 5, 36, 49, 66, 67, 84-89.

Algebraic Reynolds Stress Equations and Models

The Reynolds stress equation (3), with modeling represented by Eqs. (20-24), are extremely complicated to solve for a three-dimensional flow. For three-dimensional flows, there are 6 Reynolds stress transport equations, resulting in a total of 10-12 transport equations for the mean flow and turbulence quantities. This is beyond the capability of even present-day supercomputers. Hence, several attempts have been made to simplify the Reynolds stress transport equations. The simplification is aimed at reducing these PDEs to algebraic equations. Some of the approximations made are arbitrary and others are based on meaningful (and physically realistic) assumptions. The former procedure has been used successfully by many investigators to understand the complex features of the flow.

The models can be classified according to the assumptions made to reduce the "modeled" Reynolds stress equations

(PDE) to algebraic Reynolds stress equations. The author has arbitrarily classified these as follows:

1) Simplified algebraic Reynolds stress model (SARSM). Many of the modelers neglect the advection and convection terms and assume that the production of stress equals the dissipation. Such models are used mainly for understanding the physical nature of the complex flow and/or to modify the value of C_μ in the k-ϵ model.

2) Algebraic Reynolds stress model (ARSM). In this model, the net transport of stress $(\overline{u_i u_j})$ is assumed to proportional, being $\overline{u_i u_j}/k$. Since there is already a transport equation k, the Reynolds stress equations can be reduced to algebraic form.

Many of these SARSM's and ARSM's have been used to understand/interpret the complex phenomena occurring in flows with curvature, rotation, and other effects. Some of them have been used in combination with k-ϵ models to predict complex flows. Still other investigators have used them to modify the coefficients (e.g., C_μ) in the k-ϵ equation to allow for the complex interactions. All of these efforts are reviewed below.

Simplified Algebraic Reynolds Stress Model (SARSM)

As indicated earlier, the two basic assumptions made in these models are that $P=\epsilon$ and that the advection and convection terms are either zero or proportional to the production term.

One of the complex flows studied by several researchers using the SARSM, is the corner flow in ducts. The turbulence models and computation of such flows have been reported in Refs. 90 and 91. Launder and Ying[90] used these assumptions to predict flow in a square duct. The resulting SARSM equations are

$$\overline{v^2}-\overline{w^2}=C\frac{k}{\epsilon}\left(\overline{uv}\frac{\partial U}{\partial y}-\overline{wu}\frac{\partial U}{\partial z}\right)$$

$$\overline{vw}=C\frac{k}{\epsilon}\left(\overline{uw}\frac{\partial U}{\partial y}-\overline{uv}\frac{\partial U}{\partial z}\right) \tag{25}$$

where C is a constant and $(-\overline{vw})$ the streamwise stress. Comparing this expression with the algebraic eddy viscosity model, it is clear that the stress and strain vectors are not aligned and that the stresses are dependent on additional strains. Launder and Ying used the k transport equation for the velocity scale and an algebraic expression for the length scale to predict the flow in a square duct, which showed good agreement with the data.

Irwin and Smith[88] neglected the diffusion and advection terms (valid for equilibrium flows) and proved that the streamwise curvature has a major effect in the redistribution of energy and stresses. Their SARSM is given by

$$\frac{\overline{u^2}}{2k}=0.463(1+1.9F_c),\quad \frac{\overline{v^2}}{2k}=0.232(1-3.9F_c)$$

$$\frac{\overline{w^2}}{2k}=0.305,\quad \frac{\overline{uv}}{k}=0.178(1-3.2F_c),\quad F_c=\frac{U}{R}\frac{\partial U}{\partial y} \tag{26}$$

The centrifugal force is in y direction, normal to the wall. The fluctuating velocity component in this direction is v. It is clear that even a small curvature can have a major effect on the ratio of $\overline{v^2}/\overline{u^2}$ and $\overline{uv}$.

So[89] has carried out an extensive analysis of the effect of curvature using a SARSM. He utilized the models indicated earlier to provide a set of equations for curved flows and derived a simple equation for the variation of the turbulence velocity scale as a function of the Richardson number, thus presenting direct evidence for the type of correlations suggested by Eq. (8).

There have been many studies related to the effect of rotation on turbulence structure (e.g., Refs. 18, 20, 25, 60-62, 80-83, 92). Raj and Lumley[92] investigated the effect of rotation on a rotor wake. The analysis is valid for a rotor where the relative velocity is axial and the Coriolis force is acting transverse to the shear layer. Lakshminarayana and Reynolds[80] utilized the various models available for the pressure-strain and dissipation terms and, assuming that the production is equal to dissipation and neglecting the diffusion and convection terms, derived the following expression for the ratio of intensities and stresses for three-dimensional and rotating boundary-layer flows on an axial flow turbomachinery blade:

1) Stationary, three-dimensional, mild $(\Omega=0)$,

$$\frac{(-\overline{vw})}{(-\overline{uv})}=\frac{\partial W/\partial n}{\partial U/\partial n}$$

$$\frac{\overline{w^2}}{\overline{u^2}}=\frac{1.23(-\overline{vw})(\partial W/\partial n)+0.633\,(-\overline{uv})(\partial U/\partial n)}{1.23(-\overline{uw})(\partial U/\partial n)+0.633\,(-\overline{vw})(\partial W/\partial n)} \tag{27}$$

2) Rotation, three-dimensional $(W\ll\bar{U})$,

$$\frac{\overline{w^2}}{\overline{u^2}}=\frac{C_4(-\overline{uv})\partial U/\partial n+2\Omega\sin\beta(-\overline{uw})+2\Omega\cos\beta(-\overline{vw})}{C_3(-\overline{uv})\partial U/\partial n-2\Omega\sin\beta(-\overline{uw})}$$

$$\frac{(-\overline{vw})}{(-\overline{uv})}=\frac{\overline{v^2}(1-C_2)(\partial W/\partial n)-2\Omega[(-\overline{uv})\sin\beta+(\overline{w^2}-\overline{v^2})\cos\beta]}{\overline{v^2}(1-C_2)(\partial U/\partial n)+(-\overline{uw})2\Omega\cos\beta} \tag{28}$$

where C_2, C_3, and C_4, are constants and U, V, and W velocities in the streamwise s, principal normal n, and radial r directions, respectively. It is clear that both the three-dimensionality and the rotation introduce major structural changes in turbulence. Based on the information available on various correlations, Lakshminarayana and Reynolds[80] argue that the radial stresses and intensities increase with an increase in rotation for an axial turbomachinery rotor blade. These observations are consistent with the rotor wake and boundary-layer data reported in Refs. 2, 80, and 93.

Cousteix and Aupoix[82] assumed that the diffusion and advection terms in the Reynolds stress equation are small, but did not assume $P=\epsilon$. The analysis is valid for two-dimensional boundary layer in a channel, rotating about a spanwise z axis, with Coriolis force in the y direction normal to the wall (same as the Johnston et al.[3] experimental model). Their SARSM model equation is given by

$$\frac{\overline{u^2}}{k}=A+\alpha R,\quad \frac{\overline{v^2}}{k}=B-\alpha R,\quad \frac{\overline{w^2}}{k}=C$$

$$-\frac{\overline{uv}}{k}=C_\mu\frac{k^2}{\epsilon}\frac{\partial U}{\partial y}(1-\gamma R-\delta R^2) \tag{29}$$

where $R=-\Omega/(\partial U/\partial y)$, $\gamma=6.5$, and $\delta=25.3$.

Since R is negative on the leading side, $\overline{u^2}$ decreases and $\overline{v^2}$ increases as per experimental observation.[3] Furthermore, there is a substantial increase in the shear stress (on the leading side) even for small values of R.

Rotta[94] made assumptions outlined before and proved that the stresses in a three-dimensional flow are not aligned with the vector of the mean velocity gradient. His expression for the shear stress in the streamwise and cross-flow directions are given by Eq. (9).

It must be clear by now that many of the approximations made are ad-hoc and are configuration oriented. Nevertheless, the modeling effort described in this section has provided a great insight into the physics associated with the effects of centrifugal and Coriolis forces on the turbulence structure.

Algebraic Reynolds Stress Models (ARSM)

Even though the approximations made in the previous section have provided a great insight into the structure of complex flows, the models are oversimplified for use in computation. The algebraic Reynolds stress models overcome these disadvantages and relate the turbulent stresses to kinetic energy, thus avoiding the need to neglect these terms.

One of the most successful analyses is due to Rodi,[95] who provided an algebraic equation for the Reynolds stress. In this approach, the source term in the Reynolds stress equation is assumed to be proportional to the source terms in the kinetic energy equation. Galmes and Lakshminarayana[50] extended Rodi's analysis to flow with rotation and curvature. The generalized equations are presented below. In this analysis, it is assumed that

$$S(\overline{u_i u_k}) = (\overline{u_i u_k}/k)S(k)$$

where $S(\overline{u_i u_k})$ and $S(k)$ are the source terms given by Eq. (3) with modeled terms [Eqs. (20-24)]. Also,

$$\begin{aligned} S(\overline{u_i u_k}) &= -(\rho\overline{u_i u^j}U_{k,j} + \rho\overline{u_k u^j}U_{i,j}) \\ &\quad -2\rho\Omega^P(\epsilon_{ipj}\overline{u_k u^j} + \epsilon_{kpj}\overline{u_i u^j}) \\ &\quad + \text{Pressure strain term [Eq. (20)]} \\ &\quad + \text{dissipation terms [Eqs. (22) and (23)]} \end{aligned} \tag{30}$$

The source term in kinetic energy equation is given by

$$S(k) = -\rho\overline{u_i u^j}U_{i,j} - \rho\epsilon - (10/3)g^{ij}\mu k^{1/2}_{,i}k^{1/2}_{,j} \tag{31}$$

where S is (advection-diffusion) of $\overline{u_i u_j}$ or k.

Hence, the algebraic stress equation valid for flows with rotation is given by[50]

$$\frac{\overline{u_i u_k}}{k} = \frac{2}{3}\delta_{ik} + \frac{R_{ik}[1-(C_2/2)F] + (P_{ik} - \tfrac{2}{3}\delta_{ik}P)(1-C_2F)}{P + \rho\epsilon^*(C_1F-1)} - \frac{\tfrac{2}{3}\rho k R_T^{-1/2}\bar{S}_{ik}}{P+\rho\epsilon^*(C_1F-1)} \tag{32}$$

where

$$\epsilon^* = \epsilon + (10/3)g^{\ell m}\nu k^{1/2}_{,\ell}k^{1/2}_{,m}$$

$$P_{ik} = -\rho(\overline{u_i u^j}U_{k,j} + \overline{u_k u^j}U_{i,j})$$

$$R_{ik} = -2\rho\Omega^P(\epsilon_{ipj}\overline{u_k u^j} + \epsilon_{kpj}\overline{u_i u^j})$$

$$C_1 = 1.5, \quad C_2 = 0.6$$

It should be remarked here that, even though the diffusion and advection of $(\overline{u_i u_k})$ do not appear explicitly, they appear implicitly through the transport equations for k and ϵ [Eqs. (11) and (12)]. Hence, the ARSM cannot be used alone, but should be employed in conjunction with k-ϵ equations. Thus, the k-ϵ/ARSM model would constitute Eqs. (1), (2), (11), (12), and (32).

Galmes and Lakshminarayana[50] assumed $P=\epsilon$ in Eq. (32) and solved the resulting algebraic equations to understand the effect of rotation and three-dimensionality on changes in the turbulence structure. Two parameters of practical relevance are three-dimensionality $[D=(\partial W/\partial y)/(\partial U/\partial y)]$ and rotation effects $(R=-2\Omega/\partial U/\partial y)$. A study of the effects of three-dimensionality indicate that an increase in the value of $|D|$ decreases the value of the $\overline{u^2}$ component and increases the value of the $\overline{w^2}$ component, with very little change in the $\overline{v^2}$ component. Similarly, $-(\overline{uv})$ decreases and $(-\overline{vw})$ increases. This redistribution of energy and stresses, brought about by the three-dimensionality, is one of the significant effects of the complex strain field. Using this model, they also investigated the effects of rotation on centrifugal and axial types of compressor blades. The effect in the former case was found to be substantial.

The ARSM has been widely used by many as a closure equation, as well as in modifying complex strain effects through the coefficient C_μ. Rodi's original ARSM is given by [in Cartesian coordinates, with $R_{ik}=0$ and wall effects neglected in Eq. (32)]

$$\frac{\overline{u_i u_j}}{k} = \frac{2}{3}\delta_{ij} + \frac{1-C_2}{C_1}\left[\frac{(P_{ij}/\epsilon) - \tfrac{2}{3}\delta_{ij}(P/\epsilon)}{1+(1/C_1)(P/\epsilon - 1)}\right] \tag{33}$$

For thin shear layers, Launder[96] has proved that Eq. (33) reduces to

$$\overline{uv} = \frac{2}{3}\,\frac{1-C_2}{C_1}\,\frac{C_1 - 1 + C_2(P/\epsilon)}{[C_1 - 1 + (P/\epsilon)]^2}\,\frac{k^2}{\epsilon}\,\frac{\partial u}{\partial y} \tag{34}$$

Equation (34) suggests that C_μ (even in the most simplified situation) is a function P/ϵ. If $P=\epsilon$, $C_\mu=$ const (0.09–0.108), which is the standard eddy viscosity (k-ϵ) model. A plot of C_μ as function P/ϵ indicates the assumption that C_μ is constant (with a value of 0.09) is valid for $P/\epsilon > 1.5$. Launder also noted the disparity between the measured values and the values derived from Eq. (34) for jets and wakes. He observed that the Reynolds stresses are associated predominantly with the largest eddies present in the flow and that k is associated with slightly smaller-scale motions. He provided an alternate model to Eq. (34), $C_\mu = f(P/\epsilon, D_k/\epsilon)$, where D_k is the diffusive gain of k. This model has been validated by Launder's group.

There have been many attempts to use the ARSMs in conjunction with the k-ϵ models for the computation of turbulent flows. These attempts may be classified as follows: 1) modified k-ϵ via C_μ from the ARSM and 2) k-ϵ/ARSM. In the former models, the value of C_μ is corrected locally through the ARSM. In the latter model k-ϵ/ARSM models are solved as coupled equations.

Modified k-ϵ via C_μ from the ARSM

Some of the modifications made to the k-ϵ equation to account for the curvature and the rotation have been described earlier. The models presented in this section avoid the manipulation of empirical constants, but include C_μ as function of "extra" effects. An expression for C_μ is derived from an algebraic stress equation. These models are more successful, and rightly so, than the models based on arbitrary correction of empirical constants.

Several investigators (e.g., Refs. 49, 60, 62, 66, 67, 84, 85, 97, 98, etc.) have made attempts to modify the k-ϵ model to include the effects of curvature using ARSM. Some of these attempts are summarized below.

Pourahmadi and Humphrey[67] made a systematic and a successful attempt to modify the k-ϵ model by making C_μ [Eq. (10)] an appropriate function of the streamline curvature, while simultaneously accounting for the pressure-strain and wall-induced pressure fluctuations. The analysis employs Rodi's algebraic stress model [Eq. (33)] to derive an expression for C_μ. The analysis to some extent is similar to those employed by Gibson.[85] The analysis is more general than many other modifications proposed. The model provides good agreement with the data for two-dimensional flow with mild and strong curvature.

Pourahmadi and Humphrey[67] were able to derive an expression C_μ in terms of P, ϵ, and a velocity gradient using Eqs. (33) and (10) that resulted in the following expression:

$$C_\mu^{1/2} = 2\sqrt{Q}\cos[(I/3)\cos^{-1}(RQ^{-3/2})] - S/3 \tag{35}$$

where Q, R, and S are lengthy functions of P, ϵ, and velocity gradients.

Leschzinder and Rodi[66] derived a similar expression for C_μ given by

$$C_\mu = -K_1K_2 \Big/ \left[1 + 8K_1^2 \frac{k^2}{\epsilon^2}\left(\frac{\partial U}{\partial n} + \frac{U}{R}\right)\frac{U}{R}\right] \tag{36}$$

where y is the coordinate direction normal to the streamline.

The predictions from both these models were compared with the data from a curved channel flow.[99] The distribution of kinetic energy across the channel indicates that the model by Pourahmadi and Humphrey [Eq. (35)] shows better agreement with the data than Eq. (36). Similar conclusions have been drawn by comparison of other flows. Some of the failures and discrepancies are attributed to the departure of the assumption made in ARSMs, namely, that $\overline{u_iu_j}/k$ is constant along a streamline.

Pouagare and Lakshminarayana[62] made an attempt to develop an anisotropic model for C_μ by writing μ_t [Eq. (10)] as a vectorial quantity, with differing values in the spanwise and streamwise directions. Using a thin shear layer approximation and employing the ARSM model due to Rotta for a three-dimensional boundary layer [Eq. (9) with ν_{t1} and ν_{t2}, respectively], they proved that the coefficients in the streamwise and cross-flow directions are given, respectively, by

$$C_{\mu 1} = \alpha\left[\frac{U^2 + TW^2}{U^2 + W^2} + (1-T)\frac{UW}{U^2+W^2}\,\frac{\partial W/\partial y}{\partial U/\partial y}\right] \tag{37}$$

$$C_{\mu 2} = \alpha\left[(1-T)\frac{UW}{U^2+W^2}\,\frac{\partial U/\partial y}{\partial W/\partial y} + \frac{W^2 + TU^2}{W^2 + U^2}\right] \tag{38}$$

where $T = \tan(\gamma_t - \gamma_v)/\tan(\gamma_g - \gamma_v)$ in which γ_v, γ_t, and γ_g are angles made by the resultant velocity, shear stress, and the mean rate of strain, respectively. The predictions from this analysis are compared with the data on a rotating cylinder.[42] It should be noted that the major effect here is the three-dimensionality of the flow. Pouagare and Lakshminarayana employed k-ϵ equations (11) and (12) and Eqs. (37) and (38), a space-marching method,[100] to predict the flow on a rotating cylinder. The results[62] from the k-ϵ model and the modified (for anisotropy) model are nearly the same for the mean velocity, but the predictions of the turbulence quantities are improved considerably with these modifications.

Pouagare and Lakshminarayana also employed the ARSM of Ref. 50 for the channel rotating about a spanwise axis[3,18] and proved that (for $C_1 = 1.5$, $C_2 = 0.6$ fully developed flow)

$$-\rho\overline{uv} = \left[0.13 + 0.5\left(\frac{2\Omega}{\partial U/\partial y}\right)\right]\frac{k^2}{\epsilon}\,\frac{\partial U}{\partial y} \tag{39}$$

The agreement with Johnston's data is found to be much better when the constant 0.13 is replaced by 0.09, the value used in standard k-ϵ models. Once again, Eqs. (1), (2), (11), (12), and (39) were solved numerically using a space-marching technique. The results are compared with Johnston et al.'s data in Figs. 4 and 5. The agreement between the data and the prediction is reasonably good. It is better than Wilcox and Chambers'[49] predictions, but about the same as those of Howard et al.[64] It should be emphasized here that the prediction of Ref. 62 does not change the empirical constants used in the k-ϵ model, but rather modifies C_μ through the ARSM.

Warfield and Lakshminarayana[98] successfully implemented a modified C_μ based on the ARSM to predict the fully developed flow in a duct rotating about a spanwise axis. Assuming that the flow in the duct is two-dimensional and retaining the dominant velocity gradient $\partial U/\partial y$ (U is the streamwise velocity, y normal to wall), they assumed $F = 1$ and dropped the last term in Eq. (32) to prove that

$$C_\mu = -[2/3(C_2 - 1)(C_2P/\epsilon + C_1 - 1)]/(D_1 + D_2) \tag{40}$$

where

$$D_1 = (P/\epsilon)^2 + 2(P/\epsilon)(C_1 - 1) + (C_1 - 1)^2$$

$$D_2 = [4R_1(2 - C_2)/2]^2 + 4(C_2 - 1)(2 - C_2)R_1^2R_2$$

Here, the following natural groups appear:

$$\frac{P}{\epsilon},\quad R_1 = \frac{k\Omega}{\epsilon},\quad R_2 = \frac{\partial U/\partial y}{\Omega}$$

This relation effectively gives C_μ as a function of P/ϵ, R_1, and R_2.

The prediction of the fully developed turbulent flow in a rotating channel[3,18] was carried using a space marching Navier-Stokes code.[100] The solution was started from the channel inlet and utilized uniform inlet conditions. Figures 4 and 5 show the predictions of mean velocity, comparison with the data,[18] and comparison with the various models. It

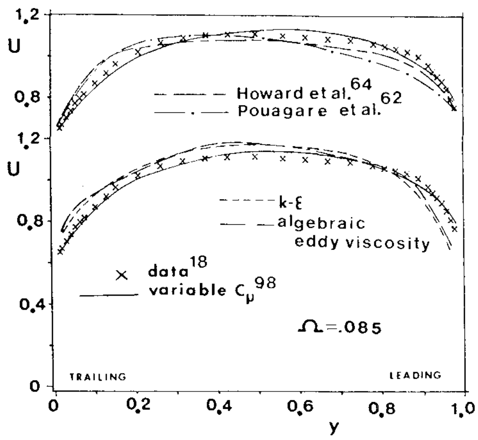


Fig. 4 Comparison between measured mean velocity and predictions in a fully developed turbulent flow in a channel rotating about a spanwise axis (U is normalized by U_{mean}).

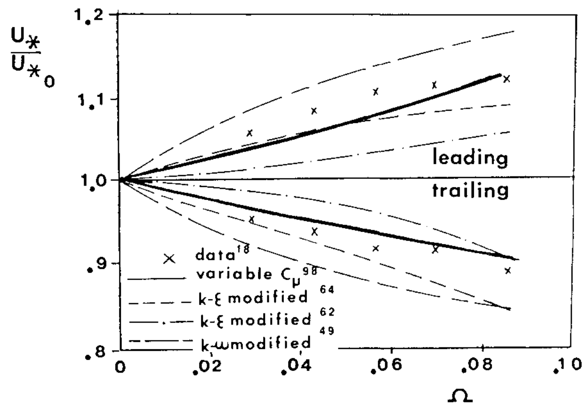


Fig. 5 Comparison between measured (or derived) skin friction velocity and prediction for a rotating duct.

is evident that both the mean velocity (Fig. 4) and the shear stress (Fig. 5) predictions from the modified C_μ model are superior to other models.

Coupled k-ϵ/ARSM's

Many attempts have been made to use k-ϵ equations (11) and (12) together with Eq. (32) or (33) to solve for $\overline{u_i u_j}$, k, and ϵ. Since the ARSM equations are algebraic, computational time is increased only moderately. Some of these attempts are described in Refs. 51, 84, 85, 88, 91, 97, and 98.

Flow with curvature (streamwise) has been computed by Gibson,[85] Irwin and Smith,[88] Rodi and Scheuerer,[84] Koosinlin and Lockwood,[97] and others. Most of these employ standard k-ϵ models [Eqs. (11) and (12) without wall function and rotation] and the ARSM [Eq. (33)]. Gibson produced an explicit form for the change in the mixing length due to curvature. He also concluded that the effects of streamline curvature on momentum and heat transfer are mainly due to an extra production term in the ARSM. These conclusions are similar to those of Irwin and Smith. Gibson also concluded that the influence of the solid wall on the turbulence is modified by the curvature. Koosinlin and Lockwood showed good agreement between predictions from k-ϵ/ARSM and the data on a rotating cylinder, rotating disk, and swirling jet. Irwin and Smith[88] showed that the curvature terms in the ARSM, although small, have a large effect. The agreement between their model and the data for the boundary layer on curved surfaces and wall jets were found to be reasonably good.

Rodi and Scheuerer[84] made a systematic comparison between the k-ϵ/ARSM model, modified k-ϵ models, and the standard k-ϵ model. The comparison, shown in Fig. 6, was made with data on the curved boundary layer by Gillis and Johnston.[101] The standard k-ϵ model[1] provides poor agreement with the C_f data. The predictions are improved with the k-ϵ model modified through the equation

$$-\overline{uv} = \left[C_\mu\left(\frac{\partial U}{\partial y} - \frac{U}{R}\right)\right]\frac{k^2}{\epsilon} \tag{41}$$

This provides a correction for C_μ, where R is the local radius (Fig. 6). For the calculation using k-ϵ/ARSM, Rodi and Scheuerer employed the ARSM derived by Gibson.[85] The comparisons show that the k-ϵ/ARSM provides a better agreement with the data, but the recovery of C_f is not predicted well. Both the modified k-ϵ model and the k-ϵ/ARSM model reduce the C_f in the curved region as measured.

Hah and Lakshminarayana[60,61] used several algebraic Reynolds stress models to predict a wide variety of complex flows, including the rotor wake, cascade wake, isolated airfoil wake, and boundary layer on a rotating hub. In this formulation, the combined effect of convection and diffusion is assumed to be proportional to the production term, resulting in the following equation, derived from Eq. (3), with the modeling for pressure strain and dissipation as indicated earlier:

$$\begin{aligned} 0 = {} & (1+C_1)(-\overline{u_k u^j}U^i_{,k} - \overline{u_k u^i}U^j_{,k})(1-\gamma) \\ & -2(\epsilon^{i\ell m}\Omega_\ell \overline{u_m u^j} + \epsilon^{j\ell m}\Omega_\ell \overline{u_m u^i}) \\ & -\tfrac{2}{3}g^{ij}\epsilon(1-\gamma) - C_{\phi 1}\frac{\epsilon}{k}(\overline{u^i u^j}) - \tfrac{2}{3}g^{ij}k \end{aligned} \tag{42}$$

The combined effect of diffusion and convection terms are related to the production term through the variable C_1. The constants used in the numerical solution are $C_{\phi 1} = 1.5$ and $\gamma = 0.6$. A standard k-ϵ equation with curvature terms appearing explicitly [Eqs. (11) and (12)] was employed in conjunction with Eq. (42). Hah and Lakshminarayana predicted the rotor wake data reported in Ref. 93. The prediction of mean velocity from both the standard k-ϵ and k-ϵ/ARSM models were found to be nearly identical. The turbulence quantities are predicted better with the use of the k-ϵ/ARSM combination (Fig. 7).

At the Stanford Conference,[1,9] Castro and Bradshaw's data for a curved jet[102] was compared with predictions from some of the ARSMs described above. The predictions from two of the models described above, the standard k-ϵ and the one-equation, were compared. The prediction of the $\overline{u^2}$ component is shown in Fig. 8. The predictions from both

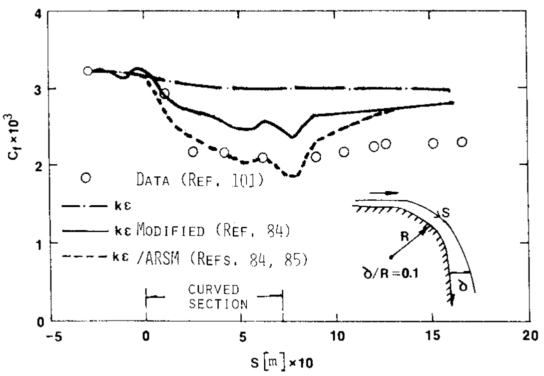


Fig. 6 Measured and predicted values of skin-friction coefficient C_f for the flow over a curved wall (Ref. 12).

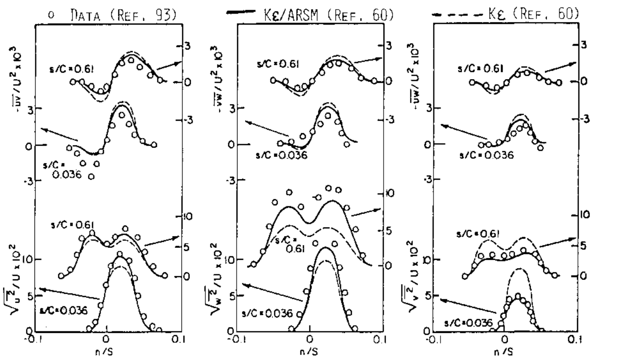


Fig. 7 Comparison between measured and predicted distribution of mean velocity, turbulence intensity, and shear stresses in a compressor rotor wake (s is the streamwise distance from the blade trailing edge, n the normal distance and zero at the trailing edge, S the blade spacing).

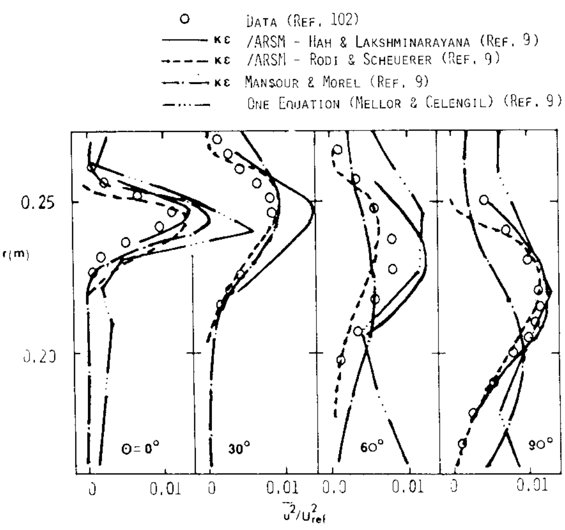


Fig. 8 Measured and predicted distribution of turbulence intensity in the principal direction of a curved wall jet[1] (r is the radius of the wall jet, θ the angular distance; see insert of Fig. 9).

k-ϵ/ARSM models are reasonably good, but the predictions from the standard k-ϵ model and the one-equation model are not good. Further comparison of this flow with various other models can be found in Refs. 9 and 84.

Baker and Orzechowski[103] have provided a nonlinear and algebraic closure model for Reynolds stress, utilizing the concepts introduced by Daly and Harlow[104] and Lumley.[72] They related the Reynolds stress to local values of k and ϵ through the equation

$$-\overline{u_i u_j} = -k\alpha_{ij} + C_4 \frac{k^2}{\epsilon} \bar{S}_{ij} + C_2 C_4 \frac{k^3}{\epsilon^2} \overline{S_{ik}} \overline{S_{kj}}$$

where $\bar{S}_{ij}$ is the symmetric mean flow strain rate tensor and α_{ij} the diagonal tensor in principal coordinates given by

$$\alpha_{ij} = (1/3k)(\overline{u_k u_k}) a_i \delta_{ij}$$

where a_i are coefficients admitting anisotropy.

Baker and Orzechowski utilized these equations along with the k-ϵ equations to provide the necessary turbulence closure. A finite element scheme was used to predict the flow in a rectangular duct and the trailing-edge flows. The predictions compare well with the data.

This model may provide a better prediction than ARSM. The nonlinear terms introduced make $\overline{u_i u_k}$ no longer locally proportional to k. However, no comparison between the two models is available at this time.

Some of the conclusions on the accuracy of the ARSM, based on the various comparisons discussed above, are as follows:

1) It is evident that the standard k-ϵ model is not adequate for the prediction of complex flows.

2) Modified k-ϵ models (modification of empirical constants) are probably adequate for very mild complex flows.

3) The most suitable models for complex flows are the models based on either the simplified ARSM or the ARSM. The k-ϵ/ARSM is probably the best model available. The k-ϵ/ARSM seems to predict the physical phenomena of complex situations. But a word of caution is in order: these models break down in situations where $\overline{u_i u_j}/k$ is not constant or when advection and diffusion terms are of substantial magnitude. An example of such a flow is that near the center of axisymmetric jets.

4) The models by Pourahmadi and Humphrey and by Gibson (k-ϵ model coupled with Rodi's ARSM) are probably among the best at the present time for predicting the curvature effects.

5) At the present time, the k-ϵ model combined with the ARSM developed by Galmes and Lakshminarayana is probably the best for predicting the effects of rotation.

In summary, k-ϵ/ARSM models simulate the turbulent stresses more realistically by relating the properties to local conditions. The k-ϵ/ARSM is efficient and inexpensive and can capture many important features of the flow. The rotation and curvature effects are captured directly instead through modeling.

Models Based on Reynolds Stress Transport Equations

The full Reynolds stress model provides a more realistic physical simulation of turbulent flow and is potentially the superior model. However, it is very complex and the least tested model so far. It is impractical for three-dimensional Navier-Stokes codes at the present time, but is suitable for the next generation of supercomputers. Nevertheless, its use is likely to become widespread during the next 5-10 years for both the simple and complex flows. Even this model does not constitute a complete closure for turbulence, as the additional physics associated with the turbulence is not simulated properly. Three types of models have been used by the computers. These are as follows:

1) Three-equation model. In this formulation, a Reynolds stress transport equation for the principal stress (e.g., $\overline{uv}$) is utilized along with the k-ϵ model [Eqs. (11) and (12)]. This provides three transport equations ($\overline{uv}$, k, and ϵ) for turbulence closure.

2) Reynolds stress models (RSM). The equations based on modeled Reynolds stress equations, including the dissipation terms [Eqs. (3), (20), (22) or (23), and (24)] are used in this model.

3) Reynolds stress plus dissipation model (RSMD). The equations based on modeled Reynolds stress equations along with a transport (PDE) equation for dissipation are used in this model. This model is designated RSMD to denote that a transport equation for dissipation is used. The most general dissipation equation (without any modeling) is given by Eq. (5). The modeled dissipation equation is given by Eq. (12). There are several variations/modifications of this model (see, e.g., Refs. 87 and 105).

The three-equation model has been employed by Biringen,[106] Durst and Rastogi[107] and Hanjalic and Launder.[77,105] In the computation used by Durst and Rastogi, the effective viscosity μ_t is not employed in the longitudinal or the principal equation. Instead, the following Reynolds stress transport equation (derived from the RSM described earlier) is used for the two-dimensional flow.

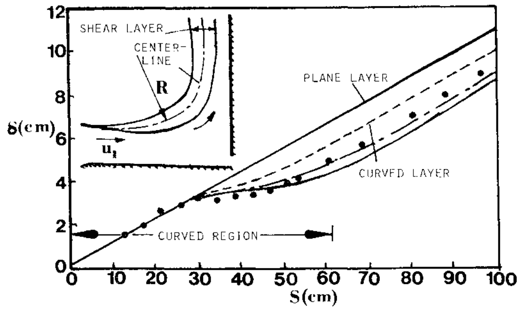
$$\frac{\partial}{\partial x}(\rho U\overline{uv}) + \frac{\partial}{\partial y}(\rho V\overline{uv}) = \frac{\partial}{\partial y}\left[\left(\frac{\mu_t}{\sigma_s} + \mu\right)\frac{\partial}{\partial y}\overline{uv} - C_s\left(\rho C_\mu k \frac{\partial U}{\partial y} + \frac{\rho\overline{uv}}{k}\epsilon\right)\right] \tag{43}$$

Biringen[106] predicted the mean velocity and the maximum stress adequately for an axisymmetric jet and wake using the three-equation model. Durst and Rastogi[107] solved the momentum equations, k-ϵ, and the modeled stress equation (43) to predict the separated flow. They utilized the k-ϵ model in the recirculating region downstream of a step. In the region downstream of the separation, the boundary-layer equations were solved with both the k-ϵ and the three-equation turbulence models. The results indicate that the three-equation model does not provide any major improvement in predictions.

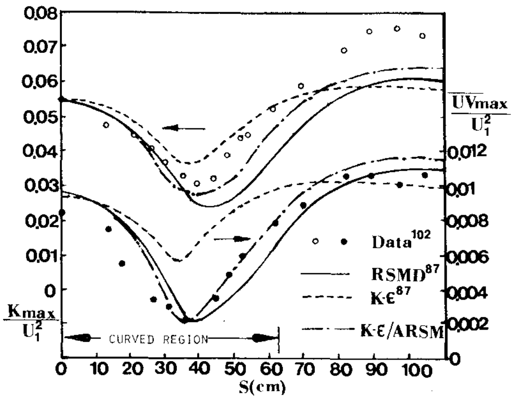
One of the early attempts to solve the flowfield without employing empirical constants is due to Lumley and Khajeh-Nouri,[73] who computed the mean and turbulence properties of an isothermal two-dimensional wake, employing the modeled Reynolds stress transport equations, a PDE for dissipation, and the mean momentum equations. Good predictions were obtained from the RSMD, except for the energy at the wake centerline.

There have been many computations of simple flows and mildly complex flow (mostly two-dimensional) using RSM's. Donaldson and Sullivan[108] predicted the decay of an isolated trailing vortex using the RSM. Reitman et al.[109] computed the incompressible flow in ducts using parabolized Navier-Stokes equations and the RSM; their predictions showed good agreement with the data. Amano and Goel[110] computed the flow in a backward-facing step using the k-ϵ, ARSM, and RSM; they found that the RSM model predicts the mean velocity profiles best in the redeveloping region, but not in the recirculating and reattachment regions. The ARSM shows better results in the latter regions. Launder and Morse[111] computed the flowfield of an axisymmetric jet. They discuss the shortcomings of various modelings used for dissipation and pressure strain correlation.

Gibson and Rodi[87] employed the RSMD and computed the flowfield in a highly curved jet measured by Castro and Bradshaw.[102] Their predictions from RSMD, ARSM, and k-ϵ



a) Growth of layer width δ.



b) Variation of maximum k and $\overline{uv}$.

Fig. 9 Comparison between predictions and data for curved mixing layer (Ref. 12).

models are compared with the data in Fig. 9. In all cases, momentum equations in a curvilinear coordinate system were employed. Both RSMD and ARSM predict the growth of the jet and maximum shear stress very well, but the recovery of $\overline{k^2}$ downstream of the curve is not predicted adequately by any of the models. This comparison shows the major inadequacy of the k-ϵ model in predicting such flows. It is interesting to note that in this particular case, predictions from the RSM are not much superior to ARSM. This presents a dilemma for computors in the selection of models, but it should be recognized that both ARSM and RSM are approximate equations, with empirical constants and various assumptions. Launder's group[112] has recently computed the data of Castro and Bradshaw[102] using k-ϵ, ARSM, and RSM (thin layer) in an elliptic code. Their results indicate that the thin-layer Navier-Stokes equations are not entirely valid: the ARSM and RSM overestimate the damping and the k-ϵ results are reasonably good.

Gibson et al.[85-87] have employed the RSMD to predict a variety of complex flows, including flows with buoyancy effect, cuved free flow, curved wall jet, etc. The RSMD accurately predicted the location of the zero shear stress and the zero mean velocity gradient and the distance between the two. The stabilizing action of the convex surface is also predicted.

Wilcox and Rubesin[14] evaluated various models on some of the complex flows. The Reynolds stress model is found to be much superior to algebraic eddy viscosity (Marvin-Sheaffer) or the k-ω model for the experiment by Lewis et al.[113] (See Fig. 10.) This is a very severe test case on turbulence models. This is an axisymmetric boundary layer ($M=4$), subjected to favorable and adverse pressure gradients through a shaped center body. They also computed the flow on a convex surface measured by So and Mellor.[114] The results indicate that the RSM is only slightly superior to those of the k-ω model.

In summary, it is not clear under what situations the RSM/RSMDs are superior to the k-ϵ/k-ω models. For very complex flows, such as those predicted by Gibson, Wilcox, and others, the RSM/RSMD may be much superior. But for moderate-to-simple shear layers, the k-ϵ or k-ω model may be adequate.

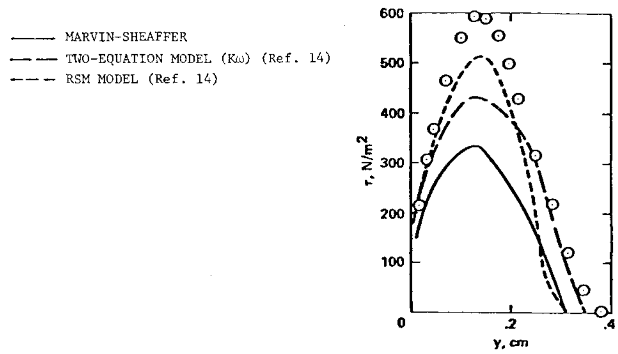


Fig. 10 Predicted and measured turbulent shear stress profile for the bump on an axisymmetric cylinder.[14]

There has been some discussion on the applicability of second-order closure models as it relates to satisfaction of material indifference, which would require that the closure relations tying the Reynolds stress tensor to the mean velocity field be invariant under change of frame.[115,116] Lumley[15] asserts that the material indifference is not applicable to turbulent flows. But Speziale's[115,116] analysis raises some questions and deserves attention.

Conclusions and Recommendations

It is evident from this review that the models, with a constant value of C_μ, are not adequate for the prediction of complex shear layers. For two-dimensional flows with separation, curvature, or rotation, a two-equation model with a proper expression for C_μ (as a function of P, ϵ, and mean velocity gradients) may be adequate. The expression for C_μ can be derived from the ARSM. The RSM/RSMD's should be employed for cases with very severe extra strain, large separation, curvature, or rotation effects. The change in turbulence structure comes from both the direct and indirect contributions from the pressure-strain correlation.

For three-dimensional flows with curvature, rotation, or shock separation, the k-ϵ equation should be coupled with either an ARSM or RSM to provide adequate prediction of the mean and the turbulence flowfield. If the interest lies in only the mean quantities, the k-ϵ model modified to include a vectorial representation of C_μ (through the ARSM) may be adequate.

For cases where shock-induced separation exists, none of the present models will predict the turbulence amplification through the shock. Future research should be directed toward resolving this inadequacy in turbulence models.

Many of the computors employ slip conditions, for both turbulence and mean velocity, and this may mask many of the deficiencies in the turbulence modeling or the code. It is desirable to determine the wall conditions from the ARSM.

A word of caution is in order regarding the performance of turbulence models. As indicated by Launder and others in Ref. 1, the discrepancy between the data and predictions may well be due to the numerical techniques employed, grid size, wall conditions, and other numerical effects. An attempt should be made to test various turbulence models, while keeping all other numerical simulation identical, to determine the performance of various turbulence models. A corollary, however, might be a caution to have redundancy in experimental measurement or at least an error analysis to provide computors with guidelines for comparison.

None of these models have been tested on very complex flows, such as annulus wall boundary-layer development in jet engines, where the viscous layer is subjected to sudden application and removal of strain through series of blade rows.

It is recommended that systematic and basic experimental investigations be carried out to isolate various complex interactions that exist in flows with curvature, rotation, separation, and vortex. This work is essential before a reasonable effort can be made to model the various terms in the Reynolds stress equations.

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